Indices – summary and practice

The basic rules

Multiplication

The rule $a^n \times a^m = a^{(n+m)}$ to *multiply* terms involving the *same* symbol, we *add* the indices.

Demonstration:
$$\begin{aligned} y^3 \times y^4 &= (y \times y \times y) \times (y \times y \times y \times y) \\ &= y \times y \times y \times y \times y \times y \times y \\ &= y^7 \\ &= y^{(3+4)} \end{aligned}$$

Example Simplify $a^2b \times ab^3$

$$\begin{aligned} a^2b \times ab^3 &= a^2 \times a \times b \times b^3 \\ &= a^2 \times a^1 \times b^1 \times b^3 \end{aligned}$$
(We could write out $a \times a \times a$ etc if we wanted to)
$$\begin{aligned} &= a^{2+1} \times b^{1+3} \\ &= a^3b^4 \end{aligned}$$

Division

The rule: $a^n \div a^m = a^{(n-m)}$ to *divide* terms involving the same symbol, we *subtract* the indices.

Demonstration:
$$2^6 \times 2^4 = \frac{2\times2\times2\times2\times2\times2}{2\times2\times2\times2}$$

$= 2 \times 2$ by cancelling the four 2s on the bottom with four of the 2s on the top
$$\begin{aligned} &= 2^2 \\ &= 2^{(6-4)} \end{aligned}$$

Example Simplify $a^2b^5 \div ab^3$
$$\begin{aligned} a^2b^5 \div ab^3 &= (a^2 \div a) \times (b^5 \div b^3) \\ &= a^{2-1} \times b^{5-3} \\ &= ab^2 \end{aligned}$$
Note This means $a(b^2)$, **not** $(ab)^2$

Exercise 1

Simplify these expressions

1. $a^5 \times a^6$ **2.** $y^3 \times y^4 \times y^5$ **3.** $3 \times 3^2 \times 3^5$ **4.** $a^5 \div a^2$

5. $2^8 \div 2^4$ **6.** $a^5 \times a^3 \div a^4$ **7.** $\frac{m^5}{m^3} \times \frac{m}{m^2}$ **8.** $\frac{aL^4}{aL^2}$

Powers of power

Without introducing anything new, we can work out what we get for a power of a power.

The rule $(a^m)^n = a^{mn}$ for terms which are a power of a power, we *multiply* the indices.

Demonstration:

$$\begin{aligned}(y^3)^2 &= (y \times y \times y)^2 \\ &= (y \times y \times y)\times(y \times y \times y) \\ &= y \times y \times y \times y \times y \times y \\ &= y^6 \\ &= y^{(3\times 2)}\end{aligned}$$

Example Simplify $(a^{23})^4$

$$(a^{23})^4 = a^{23\times 4} = a^{92}$$

Exercise 2

Simplify these expressions

1. $(a^5)^3$ **2.** $(2^3)^2$ **3.** $(ab^2)^3$ **4.** $(2x^2y^3z)^5$

Extensions from the basic rules

We know what x^n means, providing that n is a positive whole number (ie n lots of x multiplied together).

From just the simple rules we've found for multiplying and dividing power terms, we can find the meaning of other indices – ie index equal to zero, negative indices, fractional indices.

Index equal to zero

The rule: $a^0 = 1$ *anything* raised to the power 0 is 1.

Proof:

We know	$a^n \div a^m = a^{(n-m)}$	(The rule for division)
So	$a^n \div a^n = a^{(n-n)}$	(Putting $m = n$)
	$= a^0$	
but	$a^n \div a^n = 1$	(Anything (except 0) divided by itself is 1)
So	$a^0 = 1$	

Index less than zero

The rule: $a^{-n} = \dfrac{1}{a^n}$

Proof:

$$\begin{aligned}a^{-n} &= a^{(0-n)} \\ &= a^0 \div a^n \quad \text{(The rule for division)} \\ &= \frac{1}{a^n}\end{aligned}$$

Fractional indices

The rule: $a^{1/n} = \sqrt[n]{a}$ — a to the power $1/n$ is the same as the nth root of a

Demonstration: $3^{1/2} \times 3^{1/2} = 3^{(1/2 + 1/2)}$ From the basic rule for multiplying powers

$= 3^1 = 3$

But we know that the only thing you can multiply by itself to get 3 is $\sqrt{3}$, ie

$\sqrt{3} \times \sqrt{3} = 3$

So $3^{1/2} = \sqrt{3}$

Demonstration 2 $b^{1/3} \times b^{1/3} \times b^{1/3} = b^{1/3 + 1/3 + 1/3}$

$= b^1 = b$

So $b^{1/3}$ is something which, when you multiply three lots together, you get b. What is this? Answer: $\sqrt[3]{b}$

Example: Evaulate $27^{1/3}$

$27^{1/3} = \sqrt[3]{27} = 3$

Example: Write $\sqrt[3]{x^2}$ with a single factional index

$\sqrt[3]{x^2} = (x^2)^{1/3}$

$= x^{2/3}$ from the rule for powers of powers.

Example: Write $\left(\sqrt[3]{x}\right)^2$ with a single factional index

$\left(\sqrt[3]{x}\right)^2 = (x^{1/3})^2$

$= x^{2/3}$ from the rule for powers of powers.

So notice that $\left(\sqrt[3]{x}\right)^2 = \sqrt[3]{x^2}$.

Exercise 3

Evaluate these without using a calculator, then use your calculator to check the results.

1. $\left(\frac{1}{2}\right)^{-2}$ **2.** $27^{-1/3}$ **3.** $\left(\frac{1}{4}\right)^{5/2}$ **4.** $\left(\frac{100}{9}\right)^{-3/2}$ **5.** $\left(-\frac{1}{7}\right)^{-2}$

6. $3^{-1} \cdot 2^2 \cdot 4^0$ **7.** $12^{1/2} \cdot 3^{1/2}$ **8.** $27^{1/4} \cdot 3^{1/4}$ **9.** $2^4 \times 2$ **10.** $5^2 \times 5^{1/2} \times 5^{3/2}$

11. Find the values of: (a) $2^4 \times 2^2$ (b) $(10^2)^3$ (c) $(2^5)^{2/5}$ (d) $64^{1/2}$

12. Express the following as powers of 3: (a) 9^3 (b) 27^5 (c) 81^3 (d) $9^4 \times 27^3$
[Hint: $9 = 3 \times 3 = 3^2$ so $9^3 = \ldots$]

13. Express the following as powers of a: (a) $\sqrt[5]{a}$ (b) $\sqrt[3]{a^2}$ (c) $\sqrt[7]{a^4}$ (d) $\left(\sqrt{a}\right)^6$

Simplify or evaluate (as appropriate) the following:

14. $\dfrac{5^{1/3} \cdot 5^0 \cdot 25^{1/3}}{125^{1/3}}$ **15.** $\dfrac{y^{1/6} \cdot y^{-2/3}}{y^{1/4}}$ **16.** $\dfrac{p \cdot p^{-3/4}}{p^{-1/4}}$ **17.** $\dfrac{\sqrt{x} \cdot \sqrt{x^3}}{x^{-3}}$

18. $\dfrac{\left(\sqrt{t}\right)^3}{\sqrt{\left(t^5\right)}}$ **19.** $\dfrac{x^2 + x^{5/2}}{x^{-1/2}}$ **20.** $\dfrac{y^{1/2} + y^{-1/4}}{y^{-3/4}}$ **21.** $\dfrac{(x-1)^{1/2} + (x-1)^{-1/2}}{(x-1)^{1/2}}$

22. $\dfrac{m^{3/2} - m^{-1/2}}{m^{1/2} + m^{-1/2}}$ **23.** $\dfrac{3a^2 b^{3/2} c^{-4}}{12ab^2 c^{-5}}$

Answers/solutions

Exercise 1

1. a^{11}

2. $y^3 \times y^4 \times y^5 = y^{(3+4+5)} = y^{12}$

3. 3^8

4. $a^5 \div a^2 = a^{(5-2)} = a^3$

5. 2^4 (= 32)

6. $a^5 \times a^3 \div a^4 = a^{(5+3-4)} = a^4$

7. $\dfrac{m^5}{m^3} \times \dfrac{m}{m^2} = m^{5+1-3-2} = m^1 = m$

8. $\dfrac{aL^4}{aL^2} = a^{1-1}L^{4-2} = a^0 L^2 = L^2$

Exercise 2

1. $(a^5)^3 = a^{5\times 3} = a^{15}$

2. $(2^3)^2 = 2^6$

3. $(ab^2)^3 = a^3.(b^2)^3 = a^3b^6$

4. $(2x^2y^3z)^5 = 2^5.(x^2)^5.(y^3)^5.(z)^5 = 32x^{10}y^{15}z^5$

Exercise 3

1. $\left(\dfrac{1}{2}\right)^{-2} = \left(\dfrac{1}{1/2}\right)^{+2} = 2^2 = 4$

2. $27^{-1/3} = \left(\dfrac{1}{27}\right)^{+1/3} = \dfrac{1^{1/3}}{27^{1/3}} = \dfrac{1}{\sqrt[3]{27}} = \dfrac{1}{3}$

3. $\left(\dfrac{1}{4}\right)^{5/2} = \dfrac{1}{4^{5/2}} = \dfrac{1}{\sqrt[2]{4^5}} = \dfrac{1}{\sqrt{4}^5} = \dfrac{1}{2^5} = \dfrac{1}{32}$

4. $\left(\dfrac{100}{9}\right)^{-3/2} = \left(\dfrac{9}{100}\right)^{+3/2} = \dfrac{\sqrt{9}^3}{\sqrt{100}^3} = \dfrac{3^3}{10^3} = \dfrac{27}{1000}$

5. $\left(-\dfrac{1}{7}\right)^{-2} = (-7)^{+2} = (-7)\times(-7) = 49$

6. $3^{-1}\cdot 2^2 \cdot 4^0 = \dfrac{1}{3^1}\cdot 2^2 \cdot 1 = \dfrac{4}{3}$

7. $12^{1/2}\cdot 3^{1/2} = (4\times 3)^{1/2}\cdot 3^{1/2} = 4^{1/2}\cdot 3^{1/2}\cdot 3^{1/2} = \sqrt{4}\cdot 3^{1/2+1/2} = 3\sqrt{4} = 3\times 2 = 6$

8. $27^{1/4}\cdot 3^{1/4} = (3^3)^{1/4}\cdot 3^{1/4} = 3^{(3\times 1/4)+1/4} = 3$

9. $2^5 = 32$

10. $5^2 \times 5^{1/2} \times 5^{3/2} = 5^{2+1/2+3/2} = 5^4 = 625$

11. (a) $2^6 = 64$ (b) $10^{2\times 3} = 10^6 = 1\,000\,000$ (c) $2^{5\times 2/5} = 2^2 = 4$ (d) 8

12. (a) $9^3 = (3^2)^3 = 3^6$ (b) $(3^3)^5 = 3^{15}$ (c) 3^{12} (d) $3^8 \times 3^9 = 3^{17}$

13. (a) $a^{1/5}$ (b) $a^{2/3}$ (c) $a^{4/7}$ (d) $a^{6/2} = a^3$

14. $\dfrac{5^{1/3}\cdot 5^0 \cdot 25^{1/3}}{125^{1/3}} = \dfrac{5^{1/3+0}\cdot (5^2)^{1/3}}{(5^3)^{1/3}} = \dfrac{5^{1/3+2/3}}{5} = 1$

15. $\dfrac{y^{1/6}\cdot y^{-2/3}}{y^{1/4}} = y^{1/6-2/3-1/4} = y^{-3/4}$

16. 1

17. $\dfrac{\sqrt{x}\cdot\sqrt{x^3}}{x^{-3}} = x^{1/2+3/2+3} = x^5$

18. $\dfrac{(\sqrt{t})^3}{\sqrt{(t^5)}} = t^{3/2-5/2} = t^{-1} = \dfrac{1}{t}$

19. $\dfrac{x^2 + x^{5/2}}{x^{-1/2}} = \dfrac{x^2}{x^{-1/2}} + \dfrac{x^{5/2}}{x^{-1/2}} = x^{2+1/2} + x^{5/2+1/2} = x^{5/2} + x^3$

20. $\dfrac{y^{1/2} + y^{-1/4}}{y^{-3/4}} = y^{1/2+3/4} + y^{-1/4+3/4} = y^{5/4} + y^{1/2}$

21. $\dfrac{(x-1)^{1/2} + (x-1)^{-1/2}}{(x-1)^{1/2}} = \dfrac{(x-1)^{1/2}}{(x-1)^{1/2}} + \dfrac{(x-1)^{-1/2}}{(x-1)^{1/2}} = 1 + (x-1)^{-1/2-1/2} = 1 + \dfrac{1}{(x-1)} = \dfrac{(x-1)+1}{(x-1)} = \dfrac{x}{x-1}$

22. $\dfrac{m^{3/2} - m^{-1/2}}{m^{1/2} + m^{-1/2}} = \dfrac{m^{-1/2}m^2 - m^{-1/2}}{m^{-1/2}m^1 + m^{-1/2}} = \dfrac{m^2-1}{m+1} = \dfrac{(m+1)(m-1)}{m+1} = m-1$

23. $\dfrac{ab^{-1/2}c}{4} = \dfrac{1}{4}\dfrac{ac}{\sqrt{b}}$