Order of operations

Introduction ´, ¸, +, -

Suppose we wrote $4 + 2 \times 3$.

Work this out going from left to right (ie add 2 to 4 and then multiply the result by 3). What do you get?

Now work it out going from right to left (ie multiply 2 by 3, and then add the result to 4). What do you get now?

The order in which we do the calculation changes the result. Is this always the case? Try the same process on the following 'sums':

	Going Left to Right	**Going Right to Left**
1. $20 \times 4 + 2$		
2. $20 \times 4 \div 2$		
3. $20 + 4 - 2$		
4. $20 - 4 + 2$		

Can you see *why* the results are sometimes different?

In Mathematics and Science, we need everything we write to be unambiguous. We need to be sure exactly what is meant, so we need rules which always apply to tell us what order to do the calculations in.
The obvious – but *wrong!* – rule would be always to go from left to right. But life ain't that simple – there are good reasons (which we hope you'll see eventually) to have a different rule.

The first basic rules are:

- Do division and multiplication before addition and subtraction
- Do addition and subtraction after division and multiplication
- If a calculation involves division and multiplication, work from left to right
- If a calculation involves addition and subtraction, work from left to right.

eg $4 + 2 \times 3$
We always do the multiplication first, so we say $4 + \underline{2 \times 3} = 4 + \underline{6} = 10$

eg $4 - 4 \div 2 + 3 \times 2$
We do the division ($4 \div 2 = 2$) and the multiplication ($3 \times 2 = 6$) *before* we do the subtraction and addition, ie
$4 - \underline{4 \div 2} + \underline{3 \times 2} = 4 - \underline{2} + \underline{6}$.
Then, because we have a combination of addition and subtraction, work from left to right
$\underline{4 - 2} + 6 = 2 + 6 = 8$.

Exercise 1

Calculate the following without using a calculator. (Do as many as you need to so that you feel confident.)

1. $1 + 2 \times 3$	2. $2 \times 3 + 1$	3. $4 \div 2 + 1$
4. $1 + 4 \div 2$	5. $1.5 + 12 \div 3$	6. $10 \div 5 + 10 \times 2$
7. $2 + 4 - 3 \times 5 + 1$	8. $8 - 2 \times 3 + 4 \div 2 - 2 \times 3 \times 4$	

Now try typing the 'sums' into your scientific calculator. Does it give the same results? Does your calculator obey the rules for the order of operations?

Brackets

We've seen that we don't always work left to right – but instead do the multiplications and divisions first?

What happens if we want to do a calculation which ignores this rule?

The answer is to use **brackets**.

eg without brackets, $1 + 2 \times 3 = 1 + 6 = 7$
but if we want the 1 + 2 to be done first, we write

$$(1 + 2) \times 3 = (3) \times 3 = 3 \times 3 = 9$$

We can use any type of brackets we like - (2 + 4), {5-3}, [2-1]

Always work out the stuff in brackets first!

In complicated expressions, we might have more than one set of brackets. Can you see how to work out an expression like $1 + [(2 + 3) \times (2 + 5)]$?
(Think – you know you have to work out the square brackets before adding the result to the 1. How would you work out the contents of the square brackets?)

Exercise 2

Calculate the values of the following expressions (if you feel confident, do the later ones, not the earlier ones!):

1. $2 \times 3 + 1$	2. $2 \times (3 + 1)$	3. $1 + 2 \times (1 + 2) \times 3$
4. $2 + 3 - 1$	5. $3 - 2 + 1$	6. $3 - (2 + 1)$
7. $(2 \times 3) \times 5 \div 3$	8. $(2 - (3 + 1))$	9. $1 + (2 \times 3) \times 5 \div 3 - 1$
10. $3 \div ((2 - 1) \div (4 - 2))$	11. $3 \div 2 - 1 \div 4 - 2$	12. $\{40 - [2 + 1] \times [4 \times (4 - 1)]\} \div 4$

Notice that we can often add brackets *without* changing the meaning. Sometimes this is useful to help us understand what to do first:

eg $1 + 2 \times 3 = 1 + (2 \times 3) = 1 + (6) = 1 + 6 = 7$

eg $1 + 4 \div 2 - 3 \times 2 = 1 + (4 \div 2) - (3 \times 2) = \ldots$

More about symbols

If only it stopped there! There are a few extra things you need to think about:

Multiplication

Multiplication is sometimes written as * (especially in computing) and sometimes abbreviated to '·', and the symbol is sometimes missed out altogether.

eg $4\times(2+1) = 4\cdot(2+1) = 4*(2+1) = 4(2+1)$

Beware: This can cause confusion, since multiplication signs (×) look a lot like 'x', the symbol which we often use for 'an unknown amount', and '·' looks a lot like a decimal point.
As general rules:

- Make sure you write x and × differently, or if you like *always* use * instead of ×.
- Only use the · notation if it absolutely can't be mistaken for a decimal point. (eg 5.4.3.2.1 or 5·4·3·2·1 is ok as a shortcut for $5 \times 4 \times 3 \times 2 \times 1$ because it couldn't mean anything else, but only write 4·2 if you mean 4 and 2 tenths. If you want to write 4×2 then don't use the · notation.

Division

Division also has a number of different symbols:

eg $4 \div 3 = \dfrac{4}{3} = {}^{4}\!/_{3} = 4/3$

Missing brackets

When you use the horizontal line notation (eg $\frac{4}{3}$) then the line also indicates missing brackets – in other words, you calculate the stuff above the line, and the stuff below the line, before doing the division.

eg $\dfrac{7+3}{3-1} = \dfrac{(7+3)}{(3-1)} = \dfrac{10}{2} = 5$

You might notice that this is different from when we use the ÷ symbol. In the last example, if we'd written $7 + 3 \div 3 - 1$ then our earlier rule about doing division first would tell us to do this

$$7 + 3 \div 3 - 1 \quad = \quad 7 + (3 \div 3) - 1 \quad = \quad 7 + 1 - 1 = 7$$

To get something which meant the same thing, we'd need to add the missing brackets, ie

$$\frac{7+3}{3-1} = \frac{(7+3)}{(3-1)} = (7+3)/(3-1) = (7+3) \div (3-1)$$

Adding brackets

Notice in the two examples above, we *added* brackets to make the meaning clearer. It didn't alter the meaning. We can often do this – and it often helps us.

eg $1 + 2 \times 3 \quad = \quad 1 + (2 \times 3) \quad = \quad 1 + 6 = 7$

eg $\dfrac{3\times4}{6-2} = \dfrac{(3\times4)}{(6-2)} = \dfrac{12}{4} = 3$

In both these examples, the brackets aren't necessary, but they might help you to see what bits we do first.

Exercise 3

Calculate the values of the following expressions

1. $1.2.3.4$
2. $1 \times 2 \times 3 \times 4$
3. 1.2×3.4
4. $1 \times 2 \times 3 \div 4$
5. $1 \times 2 \div 3 \times 4$
6. $\dfrac{1\times2}{3\times4}$
7. $1 * 2 \times 3(4)$
8. $6 \times 12 \div 4 \div 2$
9. $6 \div 12 \times 4 \div 2$

Where could you put brackets in the following expressions without changing the meaning?

10. $1 \div 4 \times 2$
11. $2 + 3 - 5$
12. $5 - 24.5 \times 32.45 / 21$
13. $2 + 3 \times 4 / 5 - 3 \div 8.1 + 2.4.6.8$

Indices

'Indices' is the plural form for 'index'. It's a technical name for powers.

Powers are things like 3^2. We say that '3 is raised to the power 2' or that 3 has index 2.

If you're not sure about how to use indices, then there's another sheet about them.

We're mentioning them here, because they're also important when we decide what order to do things in.

Work out indices/powers* after *brackets but* before *everything else

eg $3^2 + 1 \quad = \quad (3^2) + 1 \quad = \quad 9 + 1 \quad = \quad 10$

$(2 + 3)^2 \quad = \quad (5)^2 \quad = \quad 5^2 \quad = \quad 25$

Remembering the order

The order we do calculations is best remembered using **BIDMAS**

B rackets	ALWAYS do **brackets** first.
I ndices	Do **indices** (powers) after brackets
D ivision	**Division** and **multiplication**
M ultiplication	come together
A ddition	**Addition** and **subtraction**
S ubtraction	come together

(You might have seen this before in a slightly different form: 'BODMAS' stands for **b**rackets, **o**f, **d**ivision, **m**ultiplication, **a**ddition, **s**ubtraction. I don't like this – we almost never talk about 'of' as an operation, and it misses out indices.)

Calculators

Scientific calculators obey the rules of BIDMAS.

eg if you type [1] [+] [2] [×] [3] [=]

then your calculator should work out $1 + (2 \times 3) = 7$

Check that your calculator does this.

If you wanted to work out $(1 + 2) \times 3$, how would you do this with a calculator?

Exercise 4

Calculate the following by hand, and try using your calculator to check that it obeys 'BIDMAS'.

1. $2 + (3 - 2)^2 \times 4$
2. $26 - 4(10 - 2^2)$
3. $8 - 3 \times 2.4$
4. $\dfrac{8*(5-2)}{6}$
5. $4^2[3+2(5-2)-1]$
6. $\left(\dfrac{2\cdot4\cdot6\cdot(10-2)}{24/2} - 3\right)\times\left(\dfrac{4}{2}+1\right)^2$

Answers

Exercise 1

1. 7 **2.** 7 **3.** 3 **4.** 3 **5.** 5.5

6. 22 **7.** $2 + 4 - 3 \times 5 + 1 = 2 + 4 - 15 + 1 = -8$

8. $8 - \underline{2 \times 3} + \underline{4 \div 2} - \underline{2 \times 3 \times 4} = 8 - 6 + 2 - 24 = -20$

Exercise 2

1. 7 **2.** 8 **3.**

$1 + 2 \times (1 + 2) \times 3 = 1 + 2 \times 3 \times 3 = 1 + (2 \times 3 \times 3) = 19$

4. 4 5. 2 6. 0 7. 10 8. −2

9. 10 10. 6 11.

$$3 \div 2 - 1 \div 4 - 2 = \frac{3}{2} - \frac{1}{4} - 2 = \frac{6-1-8}{4} = -\frac{3}{4}$$

10. $\{40 - [2+1]\times[4\times(4-1)]\} \div 4 = \{40 - [3]\times[4\times(3)]\} \div 4 = \{40 - 36\} \div 4 = \{4\} \div 4 = 1$

Exercise 3

1. 24 **2.** 24 **3.** 24 **4.** 3/2 = 1.5 **5.** 8/3

6. 1/6 **7.** 24 **8.** 9 **9.** 1

10. $((1 \div 4) \times 2)$ 11. $((2+3)-5)$ note that $(2 + (3 - 5))$ has the same value, but changes the order of the calculation.

12. {5 − [(24.5 × 32.45) / 21] } **{**5 − [24.5 × (32.45 / 21)] } has the same value, but changes the order of the calculation.

13. {[[2 + ((3 × 4) / 5)] − (3 ÷ 8.1)] + (2.4.6.8)} (there are probably other possibilities too)

Exercise 4

1. 6 2. 2 3. 0.8 4. 4 5. 128

6. $\left(\frac{2\cdot 4\cdot 6\cdot(10-2)}{24/2}-3\right)\times\left(\frac{4}{2}+1\right)^2=\left(\frac{2\cdot 4\cdot 6\cdot 8}{12}-3\right)\times(2+1)^2=(32-3)\times 3^2=261$